

Gluon Condensates and $\overline{m}_{c,b}$ from QCD-Moments and their Ratios to Order α_s^3 and $\langle G^4 \rangle$

Stephan Narison^a

^aLaboratoire Univers et Particules de Montpellier, CNRS-IN2P3, Case 070, Place Eugène Bataillon, 34095 - Montpellier Cedex 05, France.

Abstract

We reconsider the extraction of the gluon condensates $\langle \alpha_s G^2 \rangle$, $\langle g^3 f_{abc} G^3 \rangle$ and the \overline{MS} running quark masses $\overline{m}_{c,b}$ from different $\mathcal{M}_n(Q^2)$ Moments and their Ratios by including PT corrections to order α_s^3 , NPT terms up to $\langle G^4 \rangle$ and using stability criteria of the results versus the degree n (number of Q^2 -derivative). We explicitly show that the spectral part of the lowest moment $\mathcal{M}_1(0)$ depends strongly (as expected) on its high-energy (continuum) contribution, which is minimized for $\mathcal{M}_{n \geq 3-4}(0)$. Using higher moments and the correlations of $\langle \alpha_s G^2 \rangle$ with $\langle g^3 f_{abc} G^3 \rangle$ and $\langle G^4 \rangle$, we obtain $\langle \alpha_s G^2 \rangle = (7.0 \pm 1.3) \times 10^{-2} \text{ GeV}^4$ and $\langle g^3 f_{abc} G^3 \rangle = (8.8 \pm 5.5) \text{ GeV}^2 \times \langle \alpha_s G^2 \rangle$, while our analysis favours a modified factorisation for $\langle G^4 \rangle$. Using the previous results, we re-determine $\overline{m}_c(\overline{m}_c)$ and find that the commonly used $\mathcal{M}_1(0)$ lowest moment tends to overestimate its value compared to the ones from higher moments where stable values of $\overline{m}_c(\overline{m}_c)$ versus the variations of n and the continuum models are reached. These features can indicate that the quoted errors of $\overline{m}_{c,b}$ from $\mathcal{M}_1(0)$ may have been underestimated. Our best results from different high- n moments and their ratios are: $\overline{m}_c(\overline{m}_c) = 1261(16) \text{ MeV}$ and $\overline{m}_b(\overline{m}_b) = 4171(14) \text{ MeV}$, in excellent agreement with results obtained in [1] using some judicious choices of ratios of moments.

Keywords: QCD spectral sum rules, gluon condensates, heavy quark masses.

1. Introduction

Non-zero values of the gluon condensates have been advocated by SVZ [2, 3] for non-perturbative QCD. Indeed, the gluon condensates play an important rôle in gluodynamics (low-energy theorems,...) and in some bag models as they are directly related to the vacuum energy density (with standard notations):

$$E = -\frac{\beta(\alpha_s)}{8\alpha_s^2} \langle \alpha_s G^2 \rangle. \quad (1)$$

Moreover, the gluon condensates enter in the OPE of the hadronic correlators [2] and then are important in the analysis of QCD spectral sum rules (QSSR), especially, in the heavy quarks and in the pure Yang-Mills gluonia/glueball channels where the light quark loops and quark condensates¹ are absent to leading order [4–6]. The SVZ value:

$$\langle \alpha_s G^2 \rangle \simeq 0.04 \text{ GeV}^4, \quad (2)$$

extracted (for the first time) from charmonium sum rules [2] has been challenged by different authors [4–6]. Though there are strong indications that the exact value of the gluon condensate is around (or most likely 2-3 times) this value as obtained from earlier heavy quarks $\mathcal{M}_n(Q^2)$ [7–9], FESR [10] and exponential [11] moments, heavy quark mass-splittings [14] and e^+e^- [15–19] inclusive data. Most recent determinations from τ -decay [20–22] (see however [23]) give a value

$\langle \alpha_s G^2 \rangle \simeq (0.02 \pm 0.04) \text{ GeV}^4$, while some particular choices of $\mathcal{M}_n(Q^2)$ charmonium moments give $(0.04 \pm 0.03) \text{ GeV}^4$ [24]. Lattice calculations found large range of values [25–27]. All these results indicate that the value $\langle \alpha_s G^2 \rangle$ is not yet well determined and needs to be reconsidered.

In a previous paper [1], we have extracted, for the first time within QSSR, the correlation between $\langle \alpha_s G^2 \rangle$ and $\langle g^3 f_{abc} G^3 \rangle$ by working with higher moments known to order α_s^2 and up to $\langle g^3 f_{abc} G^3 \rangle$. We have obtained:

$$g^3 f_{abc} \langle G^3 \rangle = (31 \pm 13) \text{ GeV}^2 \langle \alpha_s G^2 \rangle \quad (3)$$

or, in terms of the instanton radius:

$$\rho_c \simeq 0.98(21) \text{ GeV}^{-1} \quad (4)$$

if one uses the dilute gas instanton (DGI) model relation¹:

$$\frac{\langle g^3 f_{abc} G^3 \rangle}{\langle \alpha_s G^2 \rangle} = \frac{4}{5} \frac{12\pi}{\rho_c^2}. \quad (5)$$

One may interpret the previous value of $\langle g^3 f_{abc} G^3 \rangle$ as the one of an *effective condensate* which can absorb into it all higher dimensions condensates not accounted for when the OPE is truncated at the $D = 6$ -dimension.

In the present paper, we shall study the effects of the $D = 8$ condensates on the previous results considering the fact that these effects can be sizeable when working with higher moments [7, 9, 28]. In the same time, we shall reconsider the

Email address: snarison@yahoo.fr (Stephan Narison)

¹The heavy quark condensate contribution can be absorbed into the gluon one through the relation [2]: $\langle \bar{Q}Q \rangle = -\langle \alpha_s G^2 \rangle / (12\pi m_Q) + \dots$. An analogous relation also occurs for the mixed quark-gluon condensate [4–6].

¹Notice that estimates of ρ_c based on DGI give the range of values: 1.5 [29], 2.5 [30] and 4.5 GeV^{-1} [2].

determination of $\langle \alpha_s G^2 \rangle$ and $\overline{m}_{c,b}$ from different $\mathcal{M}_n(Q^2)$ moments and their ratios by including corrections to order α_s^3 and non-perturbative terms up to $\langle G^4 \rangle$. We shall also focus on the extraction of $\overline{m}_{c,b}$ from the widely used $\mathcal{M}_{n=1}(Q^2 = 0)$ moments.

2. Moment sum rules, stability criteria and optimal results

Here, we shall be concerned with the two-point correlator of a heavy quark $Q \equiv c, b$:

$$-(g^{\mu\nu}q^2 - q^\mu q^\nu) \Pi_Q(q^2) \equiv i \int d^4x e^{-iqx} \langle 0 | \mathcal{T} J_Q^\mu(x) (J_Q^\nu(0))^\dagger | 0 \rangle, \quad (6)$$

where : $J_Q^\mu = \bar{Q} \gamma^\mu Q$ is the heavy quark neutral vector current. Different forms of QSSR exist in the literature [4–6]. In a previous [1] and in the present paper, we work with the moments ²:

$$\begin{aligned} \mathcal{M}_n(-q^2 \equiv Q^2) &\equiv 4\pi^2 \frac{(-1)^n}{n!} \left(\frac{d}{dQ^2} \right)^n \Pi(-Q^2) \\ &= \int_{4m_Q^2}^{\infty} dt \frac{R(t, m_Q^2)}{(t + Q^2)^{n+1}}, \end{aligned} \quad (7)$$

and with their ratios:

$$r_{n/n+1}(Q^2) = \frac{\mathcal{M}_n(Q^2)}{\mathcal{M}_{n+1}(Q^2)}, \quad r_{n/n+2}(Q^2) = \frac{\mathcal{M}_n(Q^2)}{\mathcal{M}_{n+2}(Q^2)}, \quad (8)$$

where the experimental sides are more precise than the absolute moments $\mathcal{M}_n(Q^2)$.

In the following, we shall use stability criteria, i.e. a minimum dependence of the results on the variation of the finite number of derivatives n . In practice, this minimum sensitivity is signaled by the presence of a plateau or a minimum.

We shall study later the effect of the QCD continuum models on the results.

We shall denote by optimal result the one obtained within the previous stability criteria and which is less affected by the different forms of the continuum models.

3. QCD expressions of the sum rules

The QCD expressions of the moments can be derived from the ones of the vector spectral function R .

To lowest order, it reads :

$$R_{LO} = \frac{v}{2} (3 - v^2) \quad (9)$$

where $v \equiv \sqrt{1 - 4m_Q^2/t}$ is the quark velocity.

The α_s correction is known exactly to $\mathcal{O}(\alpha_s)$ [31] and an interpolating formula has been proposed in [32].

To order α_s^2 , we shall use the approximate formula given in [33] and derived from the exact expression in [34–36].

To order α_s^3 , the three lowest $\mathcal{M}_1(0)$ [37] and $\mathcal{M}_{2,3}(0)$ moments [38] are known analytically. Semi-analytic expressions

of higher moments $\mathcal{M}_n(0)$ using Padé approximants [39] and Mellin-Barnes transform [40] are also available.

The gluon condensate $\langle \alpha_s G^2 \rangle$ contribution to the two-point correlator is known to lowest order [2] and to order α_s [41].

The dimension-six condensates ($\langle g^3 f_{abc} G^3 \rangle$ and $g^4 \langle \bar{u}u \rangle^2$) contributions have been obtained by [8]. Convenient expressions for numerical analysis of different $\mathcal{M}_n(Q^2)$ moments including the $\langle g^3 f_{abc} G^3 \rangle$ term are given by [24]. We have checked some but not all of them.

The $\langle G^4 \rangle$ condensate contributions have been calculated by [28] to lowest order. The expressions of $\mathcal{M}_n(Q^2)$ have been given by [28] and [7].

In the following discussions, we shall not transcript all these previous long and tedious formulae which interested readers can find in the original papers.

4. Experimental parametrization of the sum rules

In a narrow width approximation (NWA) and for $Q \equiv c$ ³:

$$\begin{aligned} R(t) &\equiv 4\pi \text{Im} \Pi(t + i\epsilon) \\ &= \pi \frac{N}{Q_c^2 \alpha^2} \sum_{J/\psi} M_\psi \Gamma_{\psi \rightarrow e^+ e^-} \delta(t - M_\psi^2), \end{aligned} \quad (10)$$

where $N = 3$ is the colour number; M_ψ and $\Gamma_{\psi \rightarrow e^+ e^-}$ are the mass and leptonic width of the J/ψ mesons; $Q_c = 2/3$ is the charm electric charge in units of e ; $\alpha = 1/133.6$ is the running electromagnetic coupling evaluated at M_ψ^2 . We shall use the experimental values of the J/ψ parameters compiled in Table 1.

Table 1: Masses and electronic widths of the J/ψ family from PDG10 [42].

Name	Mass [MeV]	$\Gamma_{J/\psi \rightarrow e^+ e^-}$ [keV]
$J/\psi(1S)$	3096.916(11)	5.55(14)
$\psi(2S)$	3686.093(34)	2.33(7)
$\psi(3770)$	3775.2(1.7)	0.259(16)
$\psi(4040)$	4039(1)	0.86(7)
$\psi(4160)$	4153(3)	0.83(7)
$\psi(4415)$	4421(4)	0.58(7)

We shall parametrize the contributions from $\sqrt{t_c} \geq (4.6 \pm 0.1)$ GeV using either:

Model 1: The approximate PT QCD expression of the spectral function to order α_s^2 up to order $(m_c^2/t)^6$ given in [33] and the α_s^3 contribution from non-singlet contribution up to order $(m_c^2/t)^2$ given in [43].

Model 2: The asymptotic PT expression of the spectral function known to order α_s^3 where the quark mass corrections are neglected ⁴.

Model 3: Fits of different data above the $\psi(2S)$ mass: the most recent fit is done in [43] where a comparison of results from different fitting procedures can be found.

³A missprint of factor π is in [1] but does not affect the results.

⁴Original papers are given in Refs. 317 to 321 of the book in Ref. [4].

²We shall use the same normalization as [24].

5. Test of the continuum model-dependence of the moments

In this section, we test the model-dependence of the exper-

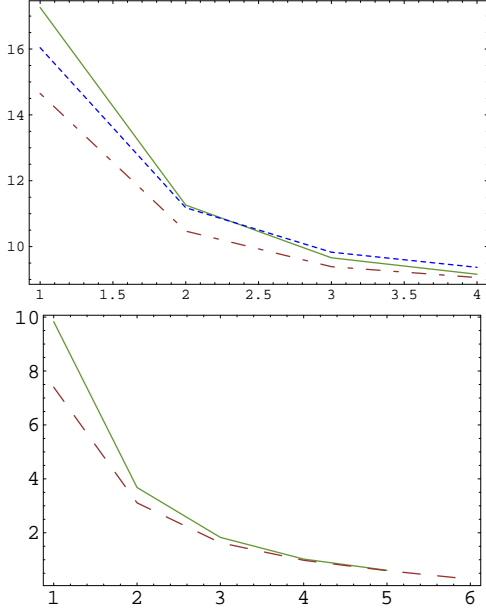


Figure 1: Behaviour of moments $\mathcal{M}_n(Q^2)$ in units of $(4\bar{m}_c^2)^n \times 10^{n+1}$ versus n for different models of the continuum as defined in section 4: **a)** $\mathcal{M}_n(0)$: Model 1: green (continuous), Model 2: red (dot-dashed), Model 3: bleu (dot); **b)** the same as Fig 1a) but for $\mathcal{M}_n(4\bar{m}_c^2)$ for Models 1 and 2.

imental side of the moments using the previous models for parametrizing the continuum (high-energy) contribution to the spectral function. The analysis is shown in Fig. 1a for the moments $\mathcal{M}_n(0)$ using Models 1, 2 and 3 for different values of n and in Fig 1b for the moments $\mathcal{M}_n(4\bar{m}_c^2)$ using Models 1 and 2. One can deduce that this model dependence can be avoided when working with values of $n \geq 3, 4$. One can also notice that for $\mathcal{M}_1(0)$, the continuum (high-energy) contribution to the moments is about (40-50)% of the total contribution, which indicates that the resulting value of m_c from the low moments $n \leq (2 - 3)$ will depend strongly on the appreciation of this high-energy behaviour which is not measured accurately as also emphasized in [43].

6. QCD inputs and higher gluon condensates

From the different expressions of the PT series given in [24], we observe that, unlike $\mathcal{M}_n(Q^2 = 0)$ where the coefficients increase approximately like n for large n (the same feature occurs for the α_s^3 term given in [38–40]), the ones of $\mathcal{M}_n(Q^2 \neq 0)$ remains (within a factor 2) almost constant though change sign from low to high moments. Therefore, we estimate the coefficient of the $O(\alpha_s^3)$ term of the moments $\mathcal{M}_n(Q^2 \neq 0)$ to be about:

$$c_3|_{Q^2 \neq 0, n} \simeq \pm c_3|_{Q^2=0, n=1} \simeq \pm 5.6, \quad (11)$$

which is larger than the estimate used in [1], where it has been assumed that the ratio of the α_s^2 over the α_s^3 coefficients are approximatively the same for each moment.

We shall use the input values [1, 19, 23]:

$$\begin{aligned} \bar{m}_c(\bar{m}_c) &= 1261(18) \text{ MeV}, \\ \alpha_s(m_c)|_{n_f=4} &= 0.408(14) \text{ from } \tau - \text{decays}, \\ \alpha_s \langle \bar{\psi}\psi \rangle^2 &= 4.5 \times 10^{-4} \text{ GeV}^6 \text{ from } e^+e^-. \end{aligned} \quad (12)$$

The error in the value of α_s is the distance between its value and the world average [42, 44].

The QCD expressions of the moments are tabulated in [24] for the fixed order PT series up to order α_s^2 and including the $\langle g^3 f_{abc} G^3 \rangle$ condensate. The contribution of the $\alpha_s^2 \langle \bar{\psi}\psi \rangle^2$ D=6 condensate is numerically negligible and has been omitted.

The contribution of the D=8 condensates can be found in [28] and [7]. In general, one can form eight operators for the D=8 gluon condensates:

$$\begin{aligned} O_1 &= \langle \text{Tr } G^2 \text{ Tr } G^2 \rangle, \\ O_2 &= \langle \text{Tr } G_{\nu\mu} G^{\rho\mu} \text{ Tr } G_{\nu\tau} G^{\rho\tau} \rangle, \\ O_3 &= \langle \text{Tr } G_{\nu\mu} G^{\tau\rho} \text{ Tr } G_{\nu\mu} G^{\tau\rho} \rangle, \\ O_4 &= \langle \text{Tr } G_{\nu\mu} G^{\tau\rho} \text{ Tr } G_{\tau\rho}^{\nu\mu} \rangle, \\ O_5 &= \langle \text{Tr } G_{\nu\mu} G^{\mu\rho} G_{\rho\tau} G^{\tau\nu} \rangle, \\ O_6 &= \langle \text{Tr } G_{\nu\mu} G^{\nu\mu} G^{\tau\rho} G_{\tau\rho} \rangle, \\ O_7 &= \langle \text{Tr } G_{\nu\mu} G^{\nu\rho} G_{\mu\tau} G^{\rho\tau} \rangle, \\ O_8 &= \langle \text{Tr } G_{\nu\mu} G^{\rho\tau} G^{\nu\mu} G^{\rho\tau} \rangle. \end{aligned} \quad (13)$$

Using the symmetry properties of the colour indices and an explicit evaluation of the trace, one can show that one has only six independent operators and the relation for $N = 3$ colours [45]:

$$\begin{aligned} O_5 + 2O_7 &= O_2 + \frac{1}{2}O_4, \\ O_8 + 2O_6 &= O_3 + \frac{1}{2}O_1. \end{aligned} \quad (14)$$

Normalized to $\langle G^2 \rangle^2$, the use of the vacuum saturation in the large N -limit gives:

$$\begin{aligned} O_1 &= \frac{1}{4} \left(1 + \frac{1}{3} \frac{1}{N^2-1} \right), & O_2 &= \frac{1}{4} \left(\frac{1}{4} + \frac{1}{3} \frac{1}{N^2-1} \right), \\ O_3 &= \frac{1}{4} \left(\frac{1}{6} + \frac{7}{6} \frac{1}{N^2-1} \right), & O_4 &= \frac{1}{4} \left(\frac{1}{12} + \frac{1}{2} \frac{1}{N^2-1} \right), \\ O_5 &= \frac{1}{4N} \left(\frac{1}{2} - \frac{1}{12} \frac{1}{N^2-1} \right), & O_6 &= \frac{1}{4N} \left(\frac{7}{6} - \frac{1}{6} \frac{1}{N^2-1} \right), \\ O_7 &= \frac{1}{4N} \left(\frac{1}{3} - \frac{1}{4} \frac{2}{N^2-1} \right), & O_8 &= \frac{1}{4N} \left(\frac{1}{3} - \frac{1}{N^2-1} \right), \end{aligned} \quad (15)$$

which indicates that only the first four operators are leading in $1/N$, and the previous constraints in Eq. (14) are not satisfied for large N . Moreover, the $1/N^2$ corrections to these leading-term are also large for $N = 3$ in the case of O_3 and O_4 , and raise some doubts on the validity of the $1/N$ -approximation. Therefore, a modified factorization has been proposed in [45], where the D=8 gluon condensates have been expressed in terms of O_2 which is not constrained. Normalized to $\langle G^2 \rangle^2$, one has:

$$\begin{aligned} O_1 &= 3O_6 = \frac{1}{4} & O_3 &= 2O_4 = -\frac{1}{16} + 2O_2 \\ O_5 &= O_7 = -\frac{1}{192} + \frac{1}{2}O_2 & O_8 &= -\frac{5}{48} + 2O_2. \end{aligned} \quad (16)$$

Ref. [45] estimates O_2 using either its large N or its factorization value. Noting that the dominant contribution to the sum rule is due to O_5 , Ref. [45] notices that the factorization proposed in [28] overestimates the $D=8$ gluon condensate contributions.

For definiteness, we use the following notations and values:

$$\begin{aligned}\rho_c &\equiv \text{instanton radius introduced in Eq.(5)}, \\ \text{fac} &= 1 \equiv \text{factorisation of } \langle G^4 \rangle, \\ \text{fac} &\simeq 0.5 \equiv \text{modified factorisation of } \langle G^4 \rangle.\end{aligned}\quad (17)$$

respectively from [28] and [45]. We also use the value of the scale $M^2 \approx 0.3 \text{ GeV}^2$ estimated in [28], which characterizes the average virtual momentum of the vacuum gluons and quarks and which relates, using factorization, some of the $D = 8$ to the $D = 6$ condensates:

$$\begin{aligned}\langle g^4 j_\mu^a D_\alpha D_\alpha j_\mu^a \rangle &= \langle g^4 j_\mu^a j_\mu^a \rangle M^2 = -\frac{4}{3} g^4 \langle \bar{u}u \rangle^2 M^2, \\ \langle g^5 f^{abc} G_{\mu\nu}^a j_\mu^b j_\nu^c \rangle &= -\frac{3}{2} g^4 \langle \bar{u}u \rangle^2 M^2, \\ \langle g^3 f^{abc} G_{\mu\nu}^a G_{\nu\lambda}^b D_\alpha D_\alpha G_{\lambda\mu}^c \rangle &= \langle g^3 f^{abc} G_{\mu\nu}^a G_{\nu\lambda}^b G_{\lambda\mu}^c \rangle M^2,\end{aligned}\quad (18)$$

where $j_\mu^a = \sum_{u,d,s} \bar{\psi} \gamma_\mu (\lambda^a/2) \psi$ and D_α the covariant derivative.

7. Hunting $\langle \alpha_s G^2 \rangle$ from higher moments $\mathcal{M}_n(Q^2)$

As mentioned in the introduction, the gluon condensate plays a key rôle in QCD gluon dynamics like is the quark condensate $\langle \bar{\psi}\psi \rangle$ for chiral symmetry breaking. We have also mentioned that there is a spread of predictions of its value in the literature. The extraction of $\langle \alpha_s G^2 \rangle$ in this paper is closed to the one using charmonium sum rules in the early literatures which follow the pioneer work of SVZ [2].

In our analysis, we shall work with higher moments which are more sensitive to $\langle \alpha_s G^2 \rangle$ but limit ourselves to the ones where the higher dimension-six and -eight condensates remain still small corrections such that the OPE remains valid. This compromise choice eliminates the use of higher $Q^2 = 0$ moments where their convergence has been the subject of hot debate in the past [8, 28, 46]. Instead the $Q^2 \neq 0$ moments converge faster [7] which allow to work with higher n -values. In the following, we shall work with $\mathcal{M}_n(Q^2 = 0)$ for $n \leq 5$, $\mathcal{M}_n(Q^2 = 4\bar{m}_c^2)$ for $n \leq 11 - 12$ and with $\mathcal{M}_n(Q^2 = 8\bar{m}_c^2)$ for $n \leq 20$ where the OPE still makes sense when using the values of the vacuum condensates given in the literature [4].

We extract $\langle \alpha_s G^2 \rangle$ using its correlations with the $D = 6$ and 8 condensates introduced above. We allow the instanton radius ρ_c which correlates $\langle \alpha_s G^2 \rangle$ and $\langle g^3 f_{abc} G^3 \rangle$ to move from 1 to 5 GeV^{-1} where the latter would be the value given by a dilute gas instanton model estimate [2]. We shall also use the relation of $\langle \alpha_s G^2 \rangle$ and $\langle g^3 f_{abc} G^3 \rangle$ with the $D=8$ condensates if one assumes a factorization hypothesis [28] or its modified form [45]. Notice that, unlike [24], we fix \bar{m}_c , which is, at present, known with good accuracy, in order to give stronger constraints on the value of $\langle \alpha_s G^2 \rangle$. We show the results as function of the number n of derivatives for $Q^2 = 4\bar{m}_c^2$ and $8\bar{m}_c^2$ and for different values of the QCD input parameters.

One can notice from the Fig 2 that the effect of α_s^3 is relatively

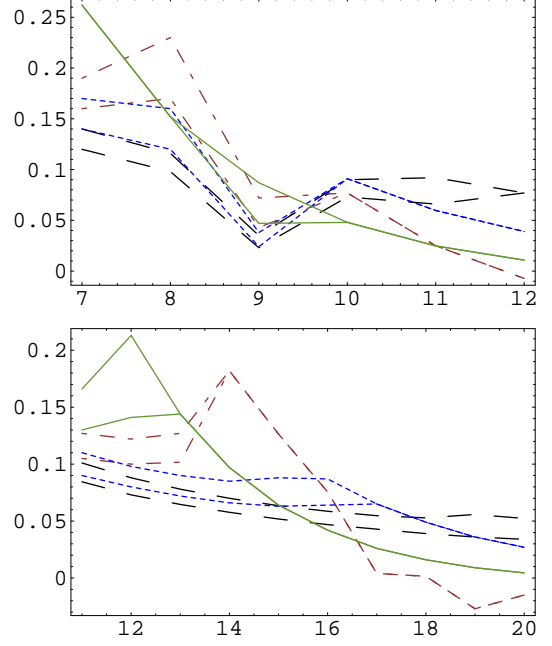


Figure 2: Behaviour of $\langle \alpha_s G^2 \rangle$ in units of GeV^4 versus n from $\mathcal{M}_n(Q^2)$ moments for different values of the QCD inputs. We use the central value of α_s given in Eq. (12). The region between the same curves correspond to the values of c_3 from Eq. (11): **a**) $\mathcal{M}_n(4\bar{m}_c^2)$: $\rho_c = 1 \text{ GeV}^{-1}$, $\text{fac}=1$: green (continuous); $\rho_c = 5 \text{ GeV}^{-1}$, $\text{fac}=1$: blue (dot); $\rho_c = 1 \text{ GeV}^{-1}$, $\text{fac}=0.5$: red (dot-dashed); $\rho_c = 5 \text{ GeV}^{-1}$, $\text{fac}=0.5$: black (dashed); **b**) The same as Fig 2a) but for $\mathcal{M}_n(8\bar{m}_c^2)$

small. Much more stable values of $\langle \alpha_s G^2 \rangle$ correspond to the case of a modified factorisation of $\langle G^4 \rangle$ which sounds better founded from the analysis of [45] based on the $1/N$ approach. Taking into account these remarks, we deduce in units of GeV^4 :

$$\begin{aligned}\langle \alpha_s G^2 \rangle &= (4.8 - 9.2) \times 10^{-2} \text{ from } \mathcal{M}_n(4\bar{m}_c^2), \\ &= (5.6 - 8.3) \times 10^{-2} \text{ from } \mathcal{M}_n(8\bar{m}_c^2),\end{aligned}\quad (19)$$

where we have used the *Mathematica Package FindRoots*, which we shall also use later on for deriving all the results in this paper.

We consider as a final result the most precise determination from $\mathcal{M}_n(8\bar{m}_c^2)$ which can be written as:

$$\langle \alpha_s G^2 \rangle = (7.0 \pm 1.3) \times 10^{-2} \text{ GeV}^4. \quad (20)$$

This result goes in line with different claims [4, 5, 8, 9, 11–14, 16, 19] that the SVZ value given in Eq. (2) underestimates the value of the gluon condensate⁵. This result agrees quite well with the one derived from the charmonium and bottomium mass-splittings using double ratio of sum rules (DRSR) [14]:

$$\langle \alpha_s G^2 \rangle = (7.5 \pm 2.5) \times 10^{-2} \text{ GeV}^4, \quad (21)$$

and from τ -like sum rule for $e^+e^- \rightarrow I = 1$ hadrons data [19]:

$$\langle \alpha_s G^2 \rangle = 6.1(0.7) \times 10^{-2} \text{ GeV}^4. \quad (22)$$

⁵A compilation of different determinations can be found in Table 2 of [14] and in the book [4] (reprinted papers in Chapters 51 and 52).

Our result is more precise than the one in [24], using some particular choice of moments, as, here, we have fixed the value of \bar{m}_c while in [24] a two-parameter fit ($\bar{m}_c, \langle \alpha_s G^2 \rangle$) has been performed. Indeed, using as input the value of $\bar{m}_c(\bar{m}_c)$ in Eq. (12), one would deduce from the different figures given in [24]:

$$\langle \alpha_s G^2 \rangle \simeq (3.5 - 7.5) \times 10^{-2} \text{ GeV}^4, \quad (23)$$

obtained to order α_s^2 and without the inclusion of $\langle G^4 \rangle$. This range of values is in agreement with the one in Eq. (19) but less precise.

8. Re-extraction of $\langle g^3 f_{abc} G^3 \rangle$ and factorisation test of $\langle G^4 \rangle$

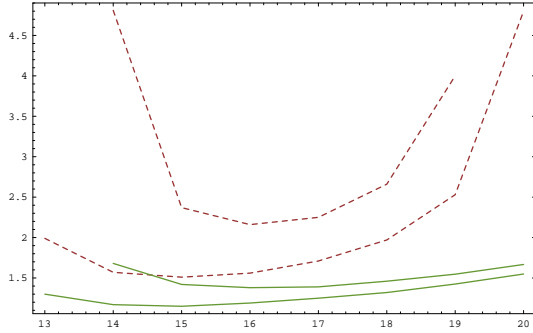


Figure 3: Behaviour of the instanton radius ρ_c in units of GeV^{-1} versus n from $\mathcal{M}_n(8m_c^2)$ moments for the central values of $\bar{m}_c(\bar{m}_c)$ and α_s given in Eq. (12). The curves correspond to $\text{fac}=0.5$ for the factorisation of $\langle G^4 \rangle$ and $\langle \alpha_s G^2 \rangle = 0.070 \text{ GeV}^4$. The region between the same curves correspond to the values of c_3 from Eq. (11): dashed (red) with $\langle G^4 \rangle$ and green (continuous) without $\langle G^4 \rangle$.

Using the previous new informations, we re-extract the value of $\langle g^3 f_{abc} G^3 \rangle$ firstly obtained in [1] using the sum rules approach. As remarked in [1], the moment $\mathcal{M}_n(8m_c^2)$ can provide the most accurate value of $\langle g^3 f_{abc} G^3 \rangle$. We plot in Fig 3 the value of the instanton radius ρ_c defined in Eq. (5) versus the number of moments for given values of \bar{m}_c , α_s , $\langle \alpha_s G^2 \rangle$ and the factorisation factor fac of the $\langle G^4 \rangle$ condensates, . We have only shown in Fig 3, the curves for $\text{fac}=0.5$ because the one for $\text{fac}=1$ gives unrealistic values of ρ_c . This can be an indirect indication that the value $\text{fac}=1$ is less favoured, a result which supports the $1/N$ result in [45]. A similar feature is also signaled when extracting $\langle \alpha_s G^2 \rangle$ because for $\text{fac}=0.5$, larger stabilities versus the change of n (see Fig 2) are obtained. At the minimas of the curves in Fig 3, we deduce the optimal value of ρ_c in GeV^{-1} when the effect of $\langle G^4 \rangle$ is included:

$$\rho_c = 1.84 \pm 0.24_{\alpha_s} \pm 0.33_{\alpha_s^3} \pm 0.27_{G^2}, \quad (24)$$

which after adding the errors quadratically gives:

$$\begin{aligned} \rho_c &= (1.84 \pm 0.49) \text{ GeV}^{-1} \\ \Rightarrow \frac{\langle g^3 f_{abc} G^3 \rangle}{\langle \alpha_s G^2 \rangle} &= (8.8 \pm 4.7) \text{ GeV}^2. \end{aligned} \quad (25)$$

We consider this result as improvement of the previous result quoted in Eq. (4), which has been affected by the presence of $\langle G^4 \rangle$ in the OPE (see the two continuous (green) curves in

Fig 3 when $\langle G^4 \rangle$ is not included ⁶) as (a priori) expected. This value of $\langle g^3 f_{abc} G^3 \rangle$ is in the range of lattice calculations in pure $SU(2)$ Yang-Mills [26], though an eventual future result for $SU(3)$ is desirable.

9. Tests of the convergence of the OPE

We show some behaviour of the OPE using the set of parameters obtained previously, namely the values of $\langle \alpha_s G^2 \rangle$ and $\langle g^3 f_{abc} G^3 \rangle$ in Eqs. (20) and (25) and the one of α_s in Eq. (12). The PT series include the coefficient $c_3 = -5.64$ of α_s^3 , while the $D=4$ condensate includes term to order α_s . Representative expressions correspond to the moments where the optimal values of $\langle \alpha_s G^2 \rangle$ and $\langle g^3 f_{abc} G^3 \rangle$ from Figs 2 and 3 are obtained. Normalized to $(4m_c^2)^n \times 10^4$, the $\mathcal{M}_{8,9}(4m_c^2)$ moments read:

$$\begin{aligned} \mathcal{M}_9(4m_c^2) &= 1.1314 \left(1.111 - \frac{0.407}{m_c^4} + \frac{0.090}{m_c^6} + \frac{0.085}{m_c^8} \right), \\ \mathcal{M}_{10}(4m_c^2) &= 0.4903 \left(1.013 - \frac{0.472}{m_c^4} + \frac{0.152}{m_c^6} + \frac{0.144}{m_c^8} \right), \end{aligned} \quad (26)$$

while the $\mathcal{M}_{15,16}(8m_c^2)$ moments normalized to $(4m_c^2)^n \times 10^9$ read:

$$\begin{aligned} \mathcal{M}_{15}(8m_c^2) &= 2.3181 \left(1.005 - \frac{0.503}{m_c^4} + \frac{0.112}{m_c^6} + \frac{0.189}{m_c^8} \right), \\ \mathcal{M}_{16}(8m_c^2) &= 0.7077 \left(0.935 - \frac{0.549}{m_c^4} + \frac{0.159}{m_c^6} + \frac{0.263}{m_c^8} \right), \end{aligned} \quad (27)$$

where one can see that the NP contributions become sizeable (the $\langle \alpha_s G^2 \rangle$ contribution is 16-22% of the LO contribution) but the OPE continues to converge (the $\langle G^4 \rangle$ contribution is less than 4%). Reciprocally, the relative large NP contributions have permitted the extraction of their size from the moments.

We also show the PT expressions of the moments normalized to $(4m_c^2)^n \times 10^4$ at fixed order:

$$\begin{aligned} \mathcal{M}_9^{PT}(4m_c^2) &= 1.1314 (1 + 0.601a_s + 2.7a_s^2 - 5.64a_s^3), \\ \mathcal{M}_{10}^{PT}(4m_c^2) &= 0.4903 (1 + 0.045a_s + 1.136a_s^2 - 5.64a_s^3), \end{aligned} \quad (28)$$

and (normalized to $(4m_c^2)^n \times 10^9$):

$$\begin{aligned} \mathcal{M}_{15}^{PT}(8m_c^2) &= 2.3181 (1 + 0.031a_s + 0.77a_s^2 - 5.64a_s^3), \\ \mathcal{M}_{16}^{PT}(8m_c^2) &= 0.7077 (1 - 0.364a_s - 0.33a_s^2 - 5.64a_s^3), \end{aligned} \quad (29)$$

where $a_s \equiv \alpha_s/\pi$. One can note that radiative corrections to these higher moments are less than 11% while it is about 30% in the case of $\mathcal{M}_0(0)$ in Eq. (31) which makes the latter sensitive to the way how the PT series is organized (fixed order, contour improved,...) as mentioned in [43].

The $D=4$ condensate contribution including the α_s corrections normalized to the LO PT moments and without the overall factor $\langle \alpha_s G^2 \rangle / (4\bar{m}_c^2)^2$ read:

$$\begin{aligned} \mathcal{M}_9^{D=4}(4m_c^2) &= -329.4(1 - 0.862a_s), \\ \mathcal{M}_{10}^{D=4}(4m_c^2) &= -433(1 - 1.673a_s), \\ \mathcal{M}_{15}^{D=4}(8m_c^2) &= -413.8(1 - 0.986a_s), \\ \mathcal{M}_{16}^{D=4}(8m_c^2) &= -491.3(1 - 1.527a_s). \end{aligned} \quad (30)$$

Again here, the α_s corrections are relatively small which is not the case of $\mathcal{M}_n(0)$ as one can see in Eq. (33).

⁶These values agree with the one obtained in [1] using some judicious choice of the ratios of moments $r_{13/14}$ and $r_{14/15}$.

10. Determinations of $\overline{m}_{c,b}$ from low moments $\mathcal{M}_{n \leq 5}(0)$

Low moments are widely used in the literature for extracting $\overline{m}_{c,b}$ where it has been argued that its QCD expression is under a good control due to the negligible contributions of NPT terms. Though this is absolutely true for $\mathcal{M}_1(0)$, the neglect of the NPT terms becomes questionable for other moments because they increase in the OPE as shown explicitly in Eq. (31). The five lowest moments normalized to $(4\overline{m}_c^2)^n$ read:

$$\begin{aligned}\mathcal{M}_1(0) &= 0.8000 \left(1.300 - \frac{0.0222}{m_c^4} + \frac{0.0005}{m_c^6} + \frac{0.001}{m_c^8} \right), \\ \mathcal{M}_2(0) &= 0.3429 \left(1.350 - \frac{0.0862}{m_c^4} + \frac{0.0076}{m_c^6} + \frac{0.007}{m_c^8} \right), \\ \mathcal{M}_3(0) &= 0.2023 \left(1.287 - \frac{0.1780}{m_c^4} + \frac{0.0368}{m_c^6} + \frac{0.027}{m_c^8} \right), \\ \mathcal{M}_4(0) &= 0.1385 \left(1.158 - \frac{0.2815}{m_c^4} + \frac{0.1172}{m_c^6} + \frac{0.077}{m_c^8} \right), \\ \mathcal{M}_5(0) &= 0.1023 \left(0.996 - \frac{0.3620}{m_c^4} + \frac{0.2959}{m_c^6} + \frac{0.184}{m_c^8} \right),\end{aligned}\quad (31)$$

which indicate that already for $n \geq 2$, one cannot neglect the non-perturbative contributions which are larger than 3.4% (compared to $\alpha_s^3 \geq 1.7\%$) in the determination of \overline{m}_c .

Another inconvenience of $\mathcal{M}_1(0)$ is the large contribution ($\geq 40\%$ effect) of the less accurate high-energy part of the spectral function which implies a model-dependent continuum contribution or a dependence on the way the non accurate data are handled as discussed explicitly in Section 5 and in [43].

Low $Q^2 = 0$ moments are also affected by large radiative corrections which one can observe from their QCD expressions given in the literature [24, 37–40]. To order α_s^3 , the PT series normalized to $(4\overline{m}_c^2)^n$ read in our normalization:

$$\begin{aligned}\mathcal{M}_1^{PT}(0) &= 0.80(1 + 2.39a_s + 2.38a_s^2 - 5.64a_s^3), \\ \mathcal{M}_2^{PT}(0) &= 0.3429(1 + 2.43a_s + 6.11a_s^2 - 7.64a_s^3), \\ \mathcal{M}_3^{PT}(0) &= 0.2032(1 + 1.92a_s + 6.12a_s^2 - 10.48a_s^3), \\ \mathcal{M}_4^{PT}(0) &= 0.1385(1 + 1.10a_s + 4.40a_s^2 - 18.13a_s^3), \\ \mathcal{M}_5^{PT}(0) &= 0.1023(1 + 0.08a_s + 2.16a_s^2 - 27.4a_s^3),\end{aligned}\quad (32)$$

where one can notice that the coefficient of a_s^3 grows with the order n of the moments, but the coefficient of a_s decreases.

The $D = 4$ contribution including the α_s corrections normalized to the lowest order PT moments and without the overall factor $\langle a_s G^2 \rangle / (4\overline{m}_c^2)^2$ read:

$$\begin{aligned}\mathcal{M}_1^{D=4}(0) &= -15.04(1 + 2.48a_s), \\ \mathcal{M}_2^{D=4}(0) &= -58.49(1 + 1.05a_s), \\ \mathcal{M}_3^{D=4}(0) &= -143.6(1 - 0.48a_s), \\ \mathcal{M}_4^{D=4}(0) &= -283.4(1 - 2.11a_s), \\ \mathcal{M}_5^{D=4}(0) &= -491.3(1 - 3.80a_s),\end{aligned}\quad (33)$$

where, one should note that one cannot go beyond $n = 5$ because the α_s correction to the $D = 4$ contribution is larger than 49% indicating the divergence of the QCD series as also emphasized by [24].

Then, we limit ourselves to use the relatively low moments $\mathcal{M}_{n \leq 5}(0)$ for extracting the running mass $\overline{m}_c(\overline{m}_c)$ within fixed order PT series and for a given set of NP parameters determined in the previous section. We show the results from the moments

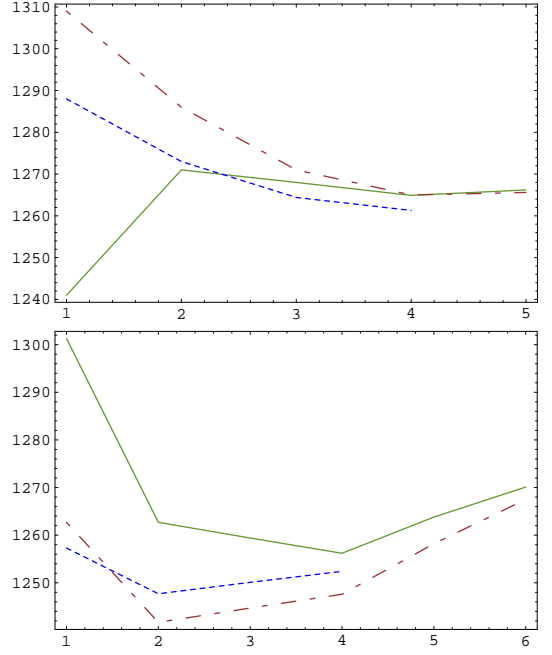


Figure 4: **a)** Behaviour of $\overline{m}_c(\overline{m}_c)$ in units of MeV versus n from $\mathcal{M}_n(0)$ moments for the central value of $\langle \alpha_s G^2 \rangle$ in Eq. (20) and of α_s given in Eq. (12). We use $\text{fac}=0.5$ for the factorisation of $\langle G^4 \rangle$. The curves correspond to different models for the continuum defined in Section 5 : Model 1: green (continued); Model 2: red (dot-dashed); Model 3 (data fit) : blue (dot). **b)** Behaviour of $\overline{m}_c(\overline{m}_c)$ versus different ratio of moments $\mathcal{M}_n(0)$ moments in units of MeV. The inputs and legends are the same as in Fig 4a). In the n -axis: $1 \equiv r_{1/2}$, $2 \equiv r_{2/3}$, $3 \equiv r_{2/4}$, $4 \equiv r_{3/4}$, $5 \equiv r_{3/5}$ and $6 \equiv r_{4/5}$

in Fig 4a) and the one from the ratios in Fig. 4b). As expected the result for $n \leq 2$ is sensitive to the Model for the continuum which contributes for 40% to the moments. One can also note that using the moments from the data fit in [43] (Model 3), the result for $n = 1$ is:

$$\overline{m}_c(\overline{m}_c)|_0^1 = 1289(8) \text{ MeV}, \quad (34)$$

where the quoted error comes only from the change in α_s given in Eq. (12) (some other sources of errors will be discussed later on). Though this result agrees with different determinations from $\mathcal{M}_1(0)$ [37, 38, 43, 47], one can note that its central value decreases when one increases the number of derivative n of the moments. The result only stabilizes versus the variation of n for $n \geq 3 - 5$, where an optimal result can be taken. For definiteness, we take $n \simeq 4$, where all *Continuum Models* give consistent predictions. Then, we deduce, from Fig 4a), in units of MeV:

$$\overline{m}_c(\overline{m}_c)|_0^4 = 1263.7 \quad (1.3)_{\text{cont}}(3.5)_{\alpha_s}(4.9)_{\alpha_s^4}(3.9)_{\mu} \quad (4.4)_{G^2}(4.7)_{G^3}(3.1)_{G^4}(1.7)_{\text{exp}}, \quad (35)$$

where the central value is the average from different continuum models. It leads to the result in Table 2.

The 1st error in Eq. (35) is due to the different models for the continuum, the 2nd one to the value of α_s given in Eq. (12). The 3rd error is an estimate of higher order terms of PT assumed to be equal to the contribution of the α_s^3 one, while the 4th error is

an estimate of the effect of the subtraction point ν by varying it from m_c to M_τ and using the substitution (see e.g. [4, 6]):

$$\alpha_s(m_c) \rightarrow \alpha_s(\nu) \times \left(1 - \beta_1 \frac{\alpha_s(\nu)}{\pi} \log \frac{\nu}{m_c}\right), \quad (36)$$

where $\beta_1 = -(1/2)(11 - 2n_f/3)$ for n_f -flavours. The 5th and 6th errors are due respectively to the ones of the gluon condensates $\langle \alpha_s G^2 \rangle$ and $\langle g^3 f_{abc} G^3 \rangle$ estimated previously. The 7th error is due to the $\langle G^4 \rangle$ condensates allowing it to move from $\text{fac}=0.5$ (favoured value from our preceeding fit) to $\text{fac}=1$ as defined in Eq. (17). The last error is due to the experimental J/ψ widths given in Table 1.

We consider as a final value the one obtained from $\mathcal{M}_4(0)$ where both PT corrections are still small for the unit and dimension 4 operators. Indeed, for the unit operator, the dominant correction is due to α_s , which is about 14% for $\mathcal{M}_4(0)$ or for m_c^2 and which is about half of the one of $\mathcal{M}_1(0)$. Then, we may expect that the error induced by the organization of the PT series (fixed order, contour improved,...) discussed in [43] is smaller for $\mathcal{M}_4(0)$ though the PT series converges faster for $\mathcal{M}_1(0)$ as one can notice in Eq. (32).

The result from the ratios of moments in Fig 4b) is not very conclusive as the model-dependence of the result starts to disappear from the ratio of moments $r_{3/5}$, but for these ratios the result increases with n . Then, we shall not retain the results from the ratios of moments for the charmonium channel.

11. $\overline{m}_c(\overline{m}_c)$ from higher $\mathcal{M}_n(Q^2)$ moments

In the following, we shall extract $\overline{m}_c(\overline{m}_c)$ from higher $\mathcal{M}_n(4m_c^2)$ and $\mathcal{M}_n(8m_c^2)$ moments. We show the results of the analysis respectively from the moments in Figs 5a) and 6a) and from the ratios of moments in Figs 5b) and 6b). One can notice that in both cases the results from the moments present minimas versus n .

The minimum is obtained from $\mathcal{M}_{10}(4m_c^2)$ and from $\mathcal{M}_{15}(8m_c^2)$, which give in units of MeV:

$$\begin{aligned} \overline{m}_c(\overline{m}_c)|_{4m_c^2}^{10} &= 1261.9 & (0.7)_{\alpha_s}(1.6)_{\alpha_s^3}(1.6)_{\alpha_s^{n \geq 4}}(0.4)_\mu \\ & & (1.1)_{G^2}(1.0)_{G^3}(1.7)_{G^4}(3.0)_{exp}, \\ \overline{m}_c(\overline{m}_c)|_{8m_c^2}^{15} &= 1260.9 & (0.5)_{\alpha_s}(1.6)_{\alpha_s^3}(1.6)_{\alpha_s^{n \geq 4}}(0.4)_\mu \\ & & (1.0)_{G^2}(0.7)_{G^3}(1.3)_{G^4}(2.6)_{exp}, \end{aligned} \quad (37)$$

which lead to the result in Table 2. The different sources of errors are the same as the ones discussed in Eq. (35). The one from α_s^3 here is due to the distance of the average of the α_s^3 contribution to the \pm assumed value of the coefficient in Eq. (11). We have estimated the error due to the unknown α_s^n ($n \geq 4$) to be equal to that of α_s^3 . Eq. (37) leads to the result in Table 2. One can also see in Figs 5b) and 6b) that the results from the ratios of moments increase with n . Though, the outputs obtained from the ratios of optimal moments are consistent with the ones from these moments and with the ones obtained in [1] where

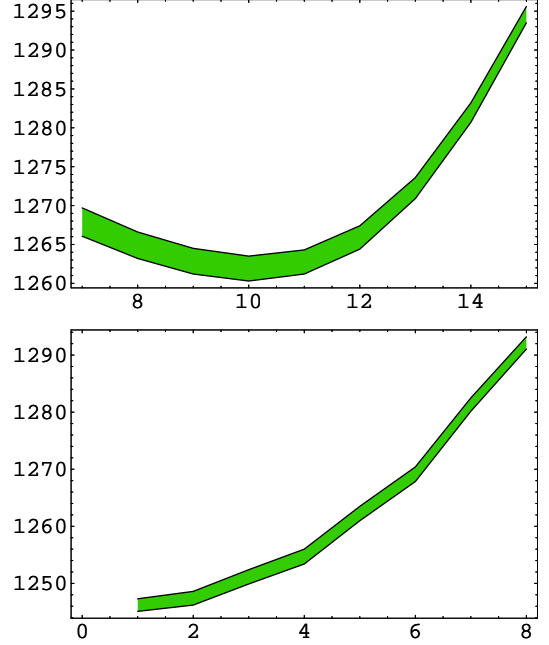


Figure 5: **a)** Behaviour of $\overline{m}_c(\overline{m}_c)$ in units of MeV versus n from $\mathcal{M}_n(4\overline{m}_c^2)$ moments and for the central value of $\langle \alpha_s G^2 \rangle$ in Eq. (20). We use $\text{fac}=0.5$ for the factorisation of $\langle G^4 \rangle$ and Model 2 for the continuum. The colored region corresponds to the range of c_3 values given in Eq. (11). **b)** Behaviour of $\overline{m}_c(\overline{m}_c)$ in units of MeV versus different ratios of moments $r_{n/n+1}(4\overline{m}_c^2)$. The inputs and legends are the same as in Fig 5a). In the n -axis: 1 $\equiv r_{7/9}$, 2 $\equiv r_{8/9}$, 3 $\equiv r_{8/10}$, 4 $\equiv r_{9/10}$...

a judicious choice (small PT corrections) of these ratios have been used, we shall not consider these numbers in the final results because of the absence of stabilities or minimas versus the variation of n .

Table 2: Value of $\overline{m}_c(\overline{m}_c)$ from charmonium moments known to α_s^3 for $Q^2 = 0$ and with an estimate of the α_s^3 contribution for $Q^2 \neq 0$.

Moments	$\overline{m}_c(\overline{m}_c)$ [MeV]
$Q^2=0$: \mathcal{M}^4	1263.7(10.3)
$Q^2=4m_c^2$: \mathcal{M}^{10}	1261.9(4.5)
$Q^2=8m_c^2$: \mathcal{M}^{15}	1260.9(4.0)

12. Final value of $\overline{m}_c(\overline{m}_c)$ and Coulombic corrections

Like in [1], we approximately estimate the Coulombic correction by working with the resummed expression of the spectral

⁷ The α_s^2 (resp α_s^3) are relatively small i.e 7.4% (resp. 3.9%).

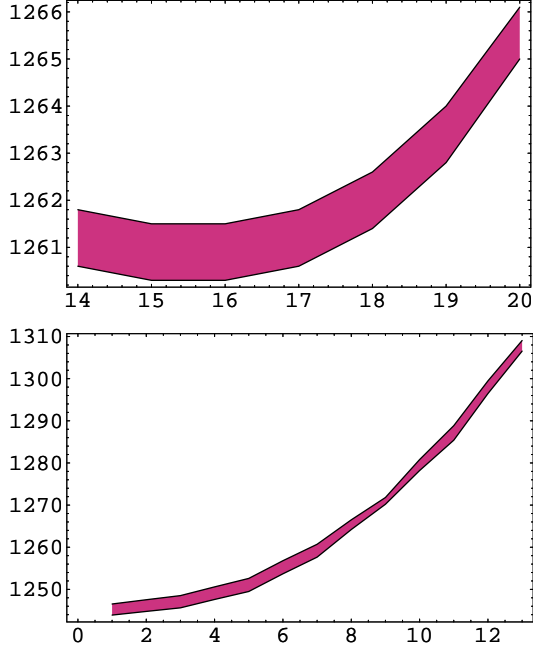


Figure 6: **a)** Behaviour of $\overline{m}_c(\overline{m}_c)$ in units of MeV versus n from $\mathcal{M}_n(8\overline{m}_c^2)$ moments for the central value of $\langle\alpha_s G^2\rangle$ in Eq. (20) and of α_s given in Eq. (12). We use $\text{fac}=0.5$ for the factorisation of $\langle G^4\rangle$ and Model 2 for the continuum. The colored region corresponds to the range of c_3 values given in Eq. (11). **b)** Behaviour of $\overline{m}_c(\overline{m}_c)$ versus different ratios of moments $\mathcal{M}_n(8\overline{m}_c^2)$ in units of MeV. The inputs and legends are the same as in Fig 6a). In the n axis: $1 \equiv r_{13/14}$, $2 \equiv r_{13/15}$, $3 \equiv r_{14/15}$, $4 \equiv r_{14/16}$, $5 \equiv r_{15/16}, \dots$

function [48]⁸:

$$\mathcal{R}_{\text{Coul}} \simeq \frac{3}{2} v \frac{x}{1 - e^{-x}}, \quad (38)$$

in an expansion in terms of $x \equiv C_F \pi \alpha_s / v$ where $C_F = 4/3$ and v is the quark velocity defined in Eq. (9). This contribution, which is of long-distance origin and proportional to the imaginary part (the wave function) of the two-point function, is induced by rescattering (Sommerfeld factor) of heavy quark pairs through the Coulomb potential above the $\bar{c}c$ threshold⁹. We add to this expression some PT QCD corrections. The 1st correction is the familiar $(1 - 4C_F \alpha_s)$ factor due to the quarkonium annihilation through a single (transverse) virtual gluon. The 2nd type of corrections to order v and $\log v$ have been obtained in [35, 36] up to order α_s^2 where the result is strictly applicable near threshold $C_F \pi \alpha_s \leq v \ll 1$ ¹⁰.

⁸However, we (a priori) expect that the non-relativistic corrections will be relatively small here as we are working in the relativistic domain because m_c is relatively light, while the final result corresponds to a relatively large $Q^2 = 8m_c^2$ value.

⁹The Coulomb corrections arising from the bound states below the threshold can be safely neglected as the dispersion relation is applied above threshold ($t \geq 4m_c^2$) where the QCD expression of the spectral function from field theory (OPE) is used while its phenomenological expression is measured from the $e^+e^- \rightarrow \text{hadrons}$ data.

¹⁰However, according to Refs. [35, 36], these short-distance effects being specific for the single annihilation process involving $Q\bar{Q}$ pairs are universal for $|v| \ll 1$ regardless whether $|v|$ is smaller or larger than $C_F \pi \alpha_s$.

We compare the value of the moments using the previous expressions for the spectral function with the one obtained from PT theory including radiative corrections up to order α_s^2 . In the case $Q^2 = 8m_c^2$ and $n = 15$, where the most precise result is obtained, the corrections induced by the previous Coulombic contributions to the value of m_c is about -1.3%^{11 12} and gives:

$$\delta_{m_c}|_{\text{Coul}} = \pm 16 \text{ MeV}. \quad (39)$$

We consider this effect as another source of error rather than a definite shift on $\overline{m}_c(\overline{m}_c)$ due to the fact that the rôle of the Coulombic effect in the sum rule analysis remains unclear [24, 49] as the quark is still relativistic with a relatively large velocity:

$$v \approx \sqrt{(1 + Q^2/4m_Q^2)/n} \simeq 0.45, \quad (40)$$

for large $n = 15$ and $Q^2 = 8m_c^2$. Indeed, this value of v would correspond to a momentum transfer between quark and anti-quark of about 1 GeV, where the effective potential differs from the Coulombic one [48] and where the sum rules are usually successfully applied.

One can see from Table 2 that the estimate from different forms of the moments are consistent each other. We shall consider as a final estimate the most precise one from $\mathcal{M}^{15}(8m_c^2)$, where the Coulombic correction obtained previously is also small. Adding this correction, we obtain:

$$\overline{m}_c(\overline{m}_c) = 1261(16) \text{ MeV}, \quad (41)$$

in excellent agreement with the one [1]:

$$\overline{m}_c(\overline{m}_c) = 1261(18) \text{ MeV}, \quad (42)$$

obtained from a judicious choice of ratios of high moments having small PT and NP corrections. The previous results also improve earlier results obtained by the author to lower orders in this channel [50].

13. Determination of $\overline{m}_b(\overline{m}_b)$

We extend the previous analysis to the bottomium systems. In the following, we shall use the value:

$$\alpha_s(m_b)|_{n_f=5} = 0.219(4), \quad (43)$$

deduced from $\alpha_s(m_\tau)$ in Eq. (12). We shall use as experimental inputs the Υ -family parameters in Table 3 using NWA and parametrize the spectral function above $\sqrt{t} = (11.098 \pm 0.079)$ GeV by its pQCD expression (Model 2), where the error in the continuum threshold is given by the total width of the $\Upsilon(11020)$. We shall work with higher moments in order to minimize the contributions of the QCD continuum. We use moments known to order α_s^3 for $Q^2 = 0$, while for $Q^2 \neq 0$, we have added the estimate of the α_s^3 contribution given in Eq. (11).

¹¹Some further arguments justifying a much smaller value of these contributions can be found in [24]. A much smaller effect of about 1 MeV is also obtained for the ratio of moments like has been found in [1].

¹²The effect on $\overline{m}_c(\overline{m}_c)$ from $\mathcal{M}^{10}(4m_c^2)$ and $\mathcal{M}^4(0)$ are respectively 2 % and 5%.

Table 3: Masses and electronic widths of the Υ family from PDG10[42].

Name	Mass [MeV]	$\Gamma_{\Upsilon \rightarrow e^+e^-}$ [keV]
$\Upsilon(1S)$	9460.30(26)	1.340(18)
$\Upsilon(2S)$	10023.26(31)	0.612(11)
$\Upsilon(3S)$	10355.2(5)	0.443(8)
$\Upsilon(4S)$	10579.4(1.2)	0.272(29)
$\Upsilon(10860)$	10865(8)	0.31(7)
$\Upsilon(11020)$	11019(8)	0.13(3)

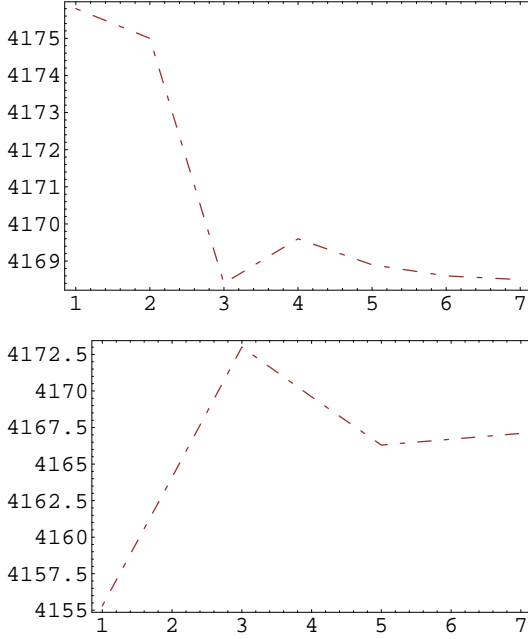


Figure 7: **a)** Behaviour of $\bar{m}_b(\bar{m}_b)$ in units of MeV versus n from $\mathcal{M}_n(0)$ moments for the central value of $\langle \alpha_s G^2 \rangle$ in Eq. (20) and of α_s given in Eq. (12). We use $\text{fac}=0.5$ for the factorisation of $\langle G^4 \rangle$ and Model 2 for the high-energy part of the spectral function. **b)** Behaviour of $\bar{m}_b(\bar{m}_b)$ versus different ratios of moments $\mathcal{M}_n(0)$ in units of MeV. The inputs and legends are the same as in Fig 7a). In the n -axis: $1 \equiv r_{2/3}$, $2 \equiv r_{2/4}$, $3 \equiv r_{3/4}$, $4 \equiv r_{3/5}$, $5 \equiv r_{4/5}, \dots$

Results from $\mathcal{M}_n(0)$

We show the results from $\mathcal{M}_n(0)$ in Fig 7, where one can notice that the result is (almost) stable versus the variation of n for $n \simeq 3 \sim 7$ while for the ratios of moments, the stability is reached from $r_{4/5}$. At these values, the contribution of the QCD continuum is less than 29% of the total which is much less than the one for $n = 1$ where it is about 66%. This feature raises serious doubts on the accurate value of m_b from this low moment $\mathcal{M}_1(0)$ given in the literature [37, 38, 47] due to the inaccuracy of the data in this high-energy region. From the moments, we obtain in units of MeV:

$$\begin{aligned} \bar{m}_b(\bar{m}_b)|_0^4 &= 4169.6 & (1.9)_{\alpha_s} (4.1)_{\alpha_s^{n \geq 4}} (2.7)_{\mu} \\ & & (1.1)_{G^2} (1.2)_{G^3} (1.2)_{G^4} (10.6)_{exp}, \\ \bar{m}_b(\bar{m}_b)|_0^{4/5} &= 4166.3 & (4)_{\alpha_s} (7.1)_{\alpha_s^{n \geq 4}} (16.6)_{\mu} \end{aligned}$$

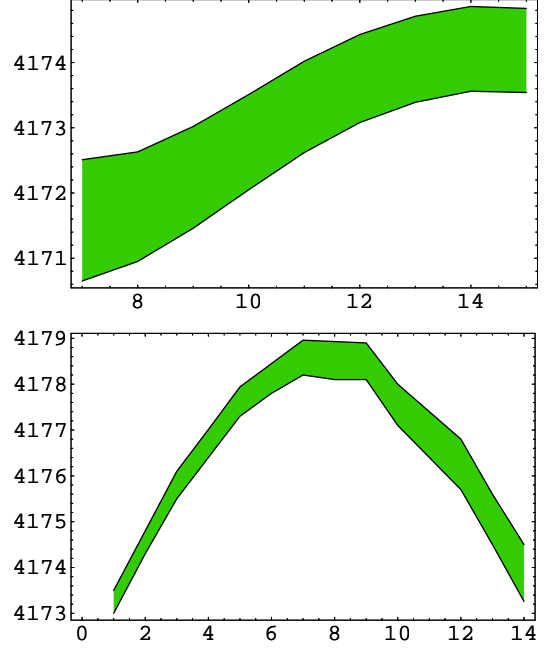


Figure 8: **a)** Behaviour of $\bar{m}_b(\bar{m}_b)$ in units of MeV versus n from $\mathcal{M}_n(4\bar{m}_b^2)$ moments for the central value of $\langle \alpha_s G^2 \rangle$ in Eq. (20) and of α_s given in Eq. (12). We use $\text{fac}=0.5$ for the factorisation of $\langle G^4 \rangle$ and Model 2 for the high-energy part of the spectral function. The colored region corresponds to the range of c_3 values given in Eq. (11). **b)** Behaviour of $\bar{m}_b(\bar{m}_b)$ versus different ratios of moments $\mathcal{M}_n(4\bar{m}_b^2)$ in units of MeV. The inputs and legends are the same as in Fig 8a). In the n -axis: $1 \equiv r_{7/8}$, $2 \equiv r_{7/9}$, $3 \equiv r_{8/9}$, $4 \equiv r_{8/10}, \dots$

$$(0.6)_{G^2} (0.4)_{G^3} (0.4)_{G^4} (1.9)_{exp}, \quad (44)$$

giving the results in Table 4. One can notice that, at the optimal choice $r_{4/5}(0)$, PT corrections are large which induce larger PT errors than in the case of $\mathcal{M}_4(0)$. The different sources of errors are similar to the case of charmonium.

Results from $\mathcal{M}_n(4m_b^2)$

The results from $\mathcal{M}_n(4m_b^2)$ are shown in Fig 8, where a stability versus the variation of n is obtained for $n = 14$, while for the ratios of moments, it is reached for $r_{10/11}(4m_b^2)$ and $r_{10/12}(4m_b^2)$. In both cases, the errors due to the NP contributions and induced by the \pm sign for the estimate of the α_s^3 coefficient and of the higher order $\alpha_s^{n \geq 4}$ are tiny (≤ 0.4 MeV) and can be neglected. We obtain in units of MeV:

$$\begin{aligned} \bar{m}_b(\bar{m}_b)|_{4m_b^2}^{14} &= 4174.2(0.6)_{\alpha_s} (2.6)_{\mu} (5.1)_{exp} \\ \bar{m}_b(\bar{m}_b)|_{4m_b^2}^{10/11} &= 4178.6(4.2)_{\alpha_s} (10.8)_{\mu} (3.6)_{exp}, \end{aligned} \quad (45)$$

from which, we deduce the result in Table 4.

Results from $\mathcal{M}_n(8m_b^2)$

The results from $\mathcal{M}_n(8m_b^2)$ and from the ratios of moments are shown in Fig 9, where stabilities versus the n -variations are respectively obtained for $n = 9 \sim 11$ and for $r_{15/17}$, $r_{16/17}$. Non perturbative corrections and the one due to the \pm sign of the α_s^3

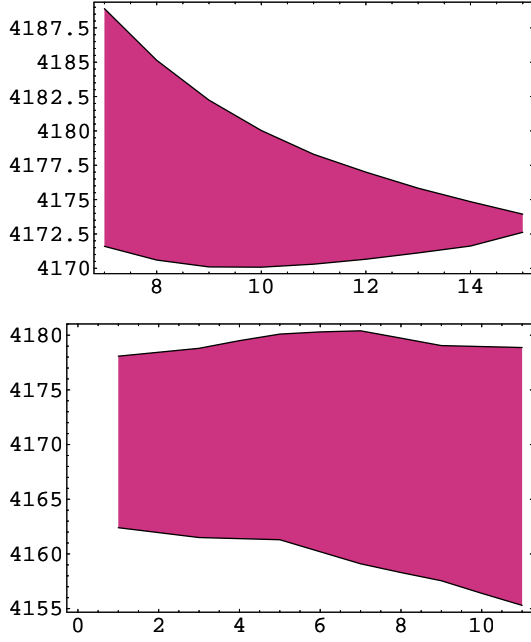


Figure 9: **a)** Behaviour of $\overline{m}_b(\overline{m}_b)$ in units of MeV versus n from $\mathcal{M}_n(8\overline{m}_b^2)$ moments for the central value of $\langle\alpha_s G^2\rangle$ in Eq. (20) and of α_s given in Eq. (12). We use $\text{fac}=0.5$ for the factorisation of $\langle G^4\rangle$. Model 2 for the high-energy part of the spectral function is used. The colored region corresponds to the range of c_3 values given in Eq. (11). **b)** Behaviour of $\overline{m}_b(\overline{m}_b)$ versus different ratios of moments $\mathcal{M}_n(8\overline{m}_b^2)$ in units of MeV. The inputs and legends are the same as in Fig 9a). In the n axis: 1 $\equiv r_{13/14}$, 2 $\equiv r_{13/15}$, 4 $\equiv r_{14/15}$, ...

coefficient are also negligible (≤ 0.3 MeV). We obtain in units of MeV:

$$\begin{aligned} \overline{m}_b(\overline{m}_b)|_{8m_b^2}^{10} &= 4175.1 & (0.5)_{\alpha_s}(5.1)_{\alpha_s^3}(0.5)_{\alpha_s^{n \geq 4}} \\ & & (1.9)_\mu(10)_{exp}, \\ \overline{m}_b(\overline{m}_b)|_{8m_b^2}^{16/17} &= 4170.9 & (1.6)_{\alpha_s}(9.5)_{\alpha_s^3}(4.1)_{\alpha_s^{n \geq 4}} \\ & & (7.2)_\mu(3.6)_{exp}. \end{aligned} \quad (46)$$

Then, we deduce the result in Table 4.

14. Final value of $\overline{m}_b(\overline{m}_b)$ and Coulombic corrections

Here, we analyze the Coulombic corrections like in the case of charm. The ones for the moments are relatively large which are respectively 1.7%, 1.1% and 4% for $\mathcal{M}^{10}(8m_b^2)$, $\mathcal{M}^{14}(4m_b^2)$ and $\mathcal{M}^4(0)$. The ones for the ratios of moments $r_{16/17}(8m_b^2)$, $r_{10/11}(4m_b^2)$ and $r_{4/5}(0)$ are respectively 1.2, 2.1 and 3.6 per mil, which are about one order of magnitude smaller. Among these different determinations the one from $r_{16/17}(8m_b^2)$ is the most precise. We consider it as our best final result:

$$\overline{m}_b(\overline{m}_b) = 4171(14) \text{ MeV}. \quad (47)$$

It is informative to compare the previous result with the one in [1] (see Table 5 from [1]) using some judicious choices of the ratios of moments having the smallest PT corrections and where the $\langle G^4\rangle$ contribution has not been included. Adding the errors ± 6 MeV due to the Coulombic, ± 6 MeV due to the subtraction

Table 4: Value of $\overline{m}_b(\overline{m}_b)$ from bottomium moments known to α_s^3 for $Q^2 = 0$ and with an estimate of the α_s^3 contribution for $Q^2 \neq 0$.

Moments	$\overline{m}_b(\overline{m}_b)$ [MeV]
$Q^2=0$:	
\mathcal{M}^4	4169.6(11.3)
$r_{4/5}$	4166.4(18.6)
$Q^2=4m_b^2$:	
\mathcal{M}^{14}	4174.2(5.8)
$r_{10/11}$	4178.6(12.1)
$Q^2=8m_b^2$:	
\mathcal{M}^{10}	4175.1(5.5)
$r_{16/17}$	4170.9(12.7)

point and ± 4 MeV due to the α_s^3 contributions, the average from Table 5 becomes:

$$\overline{m}_b(\overline{m}_b) = 4173(10) \text{ MeV}, \quad (48)$$

which is in excellent agreement with the one obtained in Eq. (47).

Table 5: Corrected values of $\overline{m}_b(m_b)$ from bottomium moments known to 3-loops using some judicious choices of moments from Ref. [1]. The errors on m_b come respectively from the choice of the moments, α_s , the data on the Υ family and the choice of the QCD continuum threshold. The ones due to the gluon condensates are negligible here.

Mom	$m_b(m_b)$ [MeV]
$Q^2=0$:	
$r_{2/3}, r_{2/4}$	4160(4)(2)(3)(3)
$Q^2=4m_b^2$:	
$r_{8/9}, r_{8/10}$	4177(2)(3)(3)(6)
$Q^2=8m_b^2$:	
$r_{13/14}, r_{13/15}$	4183(2)(4)(2)(6)
Average	4173(4)

15. Conclusions

We summarize below the main results in this letter:

We have explicitly studied in Section 5 the effect of the continuum model on the spectral function and found that this effect is large for $Q^2 = 0$ low moments, which can only be evaded for moments $\mathcal{M}_{n \geq 3-4}(Q^2)$. This feature is naturally expected but raises the question on the errors induced by this model dependence in the determinations of $\overline{m}_{c,b}$ from low-moments $\mathcal{M}_{n \leq 2}(0)$ used in the current literature.

We have extracted the value of $\langle\alpha_s G^2\rangle$ in Section 7 and found the result in Eq. (20). This result confirms previous claims that the SVZ result underestimates the value of $\langle\alpha_s G^2\rangle$. We have not included in the analysis the most eventual short distance effect of the $D = 2$ term advocated in [51–53] which is dual to

the higher order terms of the PT series [54]. However, like in different explicit analysis of some other light quark channels, the effect of this term might also be small and can improve the agreement between the QSSR predictions with the data or with some other determinations like lattice calculations. A future evaluation of this contribution is welcome but is beyond the scope of this paper.

We have re-extracted the gluon condensate $\langle g^3 f_{abc} G^3 \rangle$ and obtained its value in terms of the instanton radius ρ_c in Eq. (25). This value agrees within the error with the one in [1] but is smaller than the estimate from the DIG approximation $\rho_c \simeq 5 \text{ GeV}^{-1}$.

During the determinations of these condensates, our analysis prefers the value $\text{fac}=0.5$ of the $D = 8 \langle G^4 \rangle$ condensates which supports the modified factorisation proposed in [45] using a $1/N$ approach.

We have re-estimated the \overline{MS} running masses $\overline{m}_{c,b}$ to order α_s^3 and including the $\langle G^4 \rangle$ condensate contributions in the OPE. Optimal results from different moments lead to the final values in Eqs. (41) and (47). These results confirm the recent results in [1] obtained from judicious choices of ratios of moments with small PT corrections and where the contributions of the $D = 8 \langle G^4 \rangle$ condensates have not been included. They also improve older results in [50] obtained at lower orders with larger errors. These results are also comparable with the ones in the existing literature using different methods [24, 37, 38, 43, 47, 55, 56].

References

- [1] S. Narison, *Phys. Lett. B* **693**, (2010) 559 and Erratum-ibid. **B 705** (2011) 544.
- [2] M.A. Shifman, A.I. Vainshtein and V.I. Zakharov, *Nucl. Phys. B* **147** (1979) 385, 448.
- [3] For a review, see e.g.: V.I. Zakharov, talk given at the Sakurai's Price, *Int. J. Mod. Phys. A* **14**, (1999) 4865.
- [4] For a review, see e.g.: S. Narison, *Cambridge Monogr. Part. Phys. Nucl. Phys. Cosmol.* **17** (2004) 1-778 [hep-ph/0205006].
- [5] For a review, see e.g.: S. Narison, *World Sci. Lect. Notes Phys.* **26** (1989) 1.
- [6] For a review, see e.g.: S. Narison, *Phys. Rept.* **84** (1982) 263; S. Narison, *Acta Phys. Pol.* **B 26**(1995) 687.
- [7] L. J. Reinders, H. Rubinstein and S. Yazaki, *Phys. Lett. B* **138**, (1984) 425, *Phys. Rept.* **127** (1985) 1 and references therein.
- [8] S.N. Nikolaev and A.V. Radyushkin, *Nucl. Phys. B* **213** (1983) 285;
- [9] J. Marrow, J. Parker and G. Shaw, *Z. Phys. C* **37** (1987) 103.
- [10] F.J. Yndurain, *Phys. Rept.* **320** (1999) 287 [arXiv hep-ph/9903457].
- [11] J.S. Bell and R.A. Bertlmann, *Nucl. Phys. B* **177**, (1981) 218; *Nucl. Phys. B* **187**, (1981) 285.
- [12] R.A. Bertlmann, *Acta Phys. Austriaca* **53**, (1981) 305; *Nucl. Phys. B* **204**, (1982) 387; *Phys. Lett. B* **116**, (1982) 183; *Non-perturbative Methods*, ed. Narison, World Scientific (1985); *Nucl. Phys. (Proc. Suppl.) B* **23** (1991) 307.
- [13] R. A. Bertlmann and H. Neufeld, *Z. Phys. C* **27** (1985) 437.
- [14] S. Narison, *Phys. Lett. B* **387** (1996) 162.
- [15] G. Launer, S. Narison and R. Tarrach, *Z. Phys. C* **26** (1984) 433.
- [16] R.A. Bertlmann, G. Launer and E. de Rafael, *Nucl. Phys. B* **250**, (1985) 61 ; R.A. Bertlmann et al., *Z. Phys. C* **39**, (1988) 231.
- [17] J. Bordes, V. Gimenez, J.A. Penarrocha, *Phys. Lett. B* **201** (1988) 365.
- [18] M.B. Causse and G. Mennessier, *Z. Phys. C* **47** (1990) 611.
- [19] S. Narison, *Phys. Lett. B* **300** (1993) 293; *Phys. Lett. B* **361** (1995) 121.
- [20] The ALEPH collaboration, S. Schael et al., *Phys. Rept.* **421**, 191 (2005).
- [21] The OPAL collaboration, K. Ackerstaff et al., *Eur. Phys. J. C* **8**, 183 (1999).
- [22] M. Davier et al., *Eur. Phys. J. C* **56**, (2008) 305.
- [23] S. Narison, *Phys. Lett. B* **673** (2009) 30.
- [24] B.L. Ioffe and K.N. Zybalyuk, *Eur. Phys. J. C* **27** (2003) 229; B.L. Ioffe, *Prog. Part. Nucl. Phys.* **56** (2006) 232.
- [25] P.E. Rakow, *arXiv:hep-lat/0510046* ; G. Burgio, F. Di Renzo, G. Marchesini and E. Onofri, *Phys. Lett. B* **422** (1998) 219; R. Horley, P.E.L. Rakow and G. Schierholz, *Nucl. Phys. (Proc. Sup.) B* **106** (2002) 870.
- [26] A. Di Giacomo, *Non-perturbative Methods*, ed. Narison, World Scientific (1985); M. Campostrini, A. Di Giacomo, Y. Gunduc *Phys. Lett. B* **225** (1989) 393; A. Di Giacomo and G.C. Rossi, *Phys. Lett. B* **100** (1981) 481.
- [27] M. D'Elia, A. Di Giacomo and E. Meggiolaro, *Phys. Lett. B* **408** (1997) 315.
- [28] S.N. Nikolaev and A.V. Radyushkin, *Phys. Lett. B* **124** (1983) 243;
- [29] T. Schafer and E.V. Shuryak, *Rev. Mod. Phys.* **70** (1998) 323.
- [30] B.L. Ioffe and A.V. Samsonov, *Phys. At. Nucl.* **63** (2000) 1448.
- [31] A.O.G. Kallen and A. Sabry, *Kong. Dan. Vid.Sel. Mat. Fys. Med.* **29N17** (1955) 1.
- [32] J. Schwinger, *Particles sources and fields*, ed. Addison-Wesley Publ., **Vol 2** (1973).
- [33] K. Chetyrkin, R. Harlander, J.H. Kuhn and M. Steinhauser, *Nucl. Phys. B* **503** (1997) 339.
- [34] A. Hoang, J.H. Kuhn and T. Teubner, *Nucl. Phys. B* **452** (1995) 173.
- [35] K. Chetyrkin, J.H. Kuhn and M. Steinhauser, *Nucl. Phys. B* **482** (1996) 213.
- [36] K. Chetyrkin, A.H. Hoang, J.H. Kuhn, M. Steinhauser and T. Teubner, *Z. Phys. C* **2** (1998) 137.
- [37] K. Chetyrkin, J.H. Kuhn and C. Sturm, *Eur. Phys. C* **48** (2006) 107; R. Boughezal, M. Czakon and T. Schutzmeier, *Phys. Rev D* **74** (2006) 074006.
- [38] A. Maier, P. Mairhofer and P. Marquard, *Phys. Lett. B* **669** (2008) 88; A. Maier, P. Mairhofer, P. Marquard and A. V. Smirnov, *Nucl. Phys. B* **824** (2010) 1;
- [39] A. Hoang, V. Mateu and S.M. Zebarjad, *Nucl. Phys. B* **813** (2009) 349; Y. Kiyo, A. Maier, P. Mairhofer, P. Marquard, *Nucl. Phys. B* **823** (2009) 269;
- [40] D. Greynat and S. Peris, *Phys. Rev D* **82** (2010) 034030; arXiv:1104.3425 (2011).
- [41] D.J. Broadhurst, P.A. Baikov, V.A. Ilyin, J. Fleischer, O.V. Tarasov and V.A. Smirnov, *Phys. Lett. B* **329** (1994) 103.
- [42] PDG, C. Amsler et al., *Phys. Lett. B* **667** (2008) 1.
- [43] B. Dehnadi, A.H. Hoang, V. Mateu and S.M. Zebarjad, arXiv:1102.2264 (2011).
- [44] S. Bethke, *Prog. Part. Nucl. Phys.* **58** (2007) 351.
- [45] E. Bagan, J. I. Latorre, P. Pascual and R. Tarrach, *Nucl. Phys. B* **254** (1985) 55.
- [46] V. A. Novikov, M.A. Shifman, A.I. Vainshtein and V.I. Zakharov, *Nucl. Phys. B* **237** (1984) 525.
- [47] K.G. Chetyrkin et al., *Phys. Rev D* **80** (2009) 074010; J.H. Kuhn, M. Steinhauser and C. Sturm, *Nucl. Phys. B* **778** (2007) 192.
- [48] E. Eichten et al., *Phys. Rev D* **21** (1980) 203.
- [49] V. A. Novikov et al., *Phys. Rep.* **41** (1978) 1.
- [50] S. Narison, *Phys. Lett. B* **197** (1987) 405; *Phys. Lett. B* **216** (1989) 191; *Phys. Lett. B* **341** (1994) 73.
- [51] K. Chetyrkin, S. Narison and V.I. Zakharov, *Nucl. Phys. B* **550** (1999) 353; S. Narison and V.I. Zakharov, *Phys. Lett. B* **522**, (2001) 266; S. Narison and V.I. Zakharov, *Phys. Lett. B* **679**, (2009) 355.
- [52] For reviews, see e.g.: V.I. Zakharov, *Nucl. Phys. Proc. Suppl.* **164** (2007) 240; S. Narison, *Nucl. Phys. Proc. Suppl.* **164**, 225 (2007).
- [53] S. Narison and V.I. Zakharov, *Phys. Lett. B* **522**, (2001) 266.
- [54] S. Narison and V.I. Zakharov, *Phys. Lett. B* **679**, (2009) 355.
- [55] For a review, see e.g.: N. Brambilla et al. *Eur. Phys. J. C* **71** (2011) 1534.
- [56] I. Allison et al., *Phys. Rev. D* **78** (2008) 054513; C. McNeile et al., *Phys. Rev. D* **82** (2010) 034512.